Music and Engineering: Just and Equal Temperament

Tim Hoerning

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Outline

- Definitions and Conventions
- Notes on the Staff
- Basics of Scales
  - Harmonic Series
  - Harmonious relationships
- Cents
- Just Intonation
- Scales of Just Intonation
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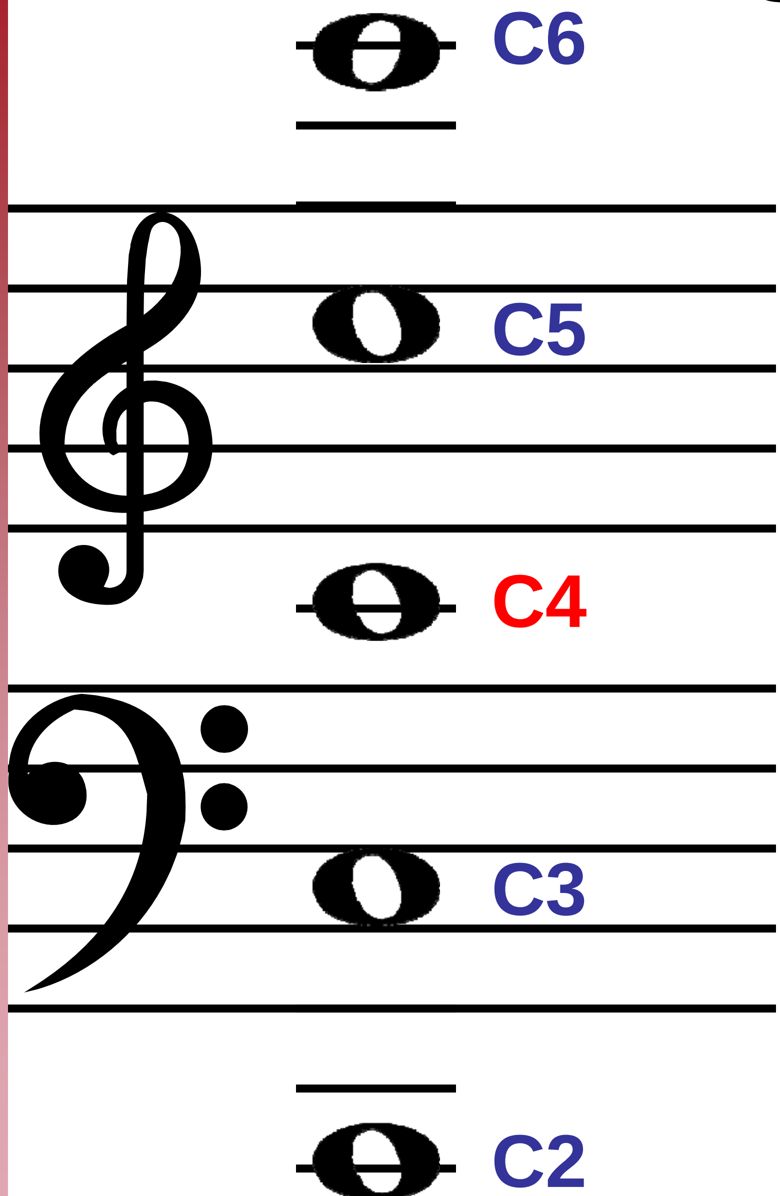
  - Major
  - Minor
- The issue: Notes ~= Frequencies
- Equal Intonation
- Scales of Equal Intonation
  - Chromatic
  - Major
  - Minor
- Even Tempered Instruments
  - Piano
  - Guitar

Definitions

- **Tone -** A tone is a sound sensation having pitch or a sound wave capable of exciting an auditory sensation having pitch.
- **Note -** A note is a conventional sign used to indicate the pitch or the duration or both of a tone sensation.
- **Overtone** - An overtone is a component of a complex tone having a pitch higher than the fundamental. An overtone is a physical component of a complex sound having a frequency higher that that of the fundamental tone.
- **Partial -** A partial is a component of a sound sensation which may be distinguished as a simple sound that cannot be further analyzed by the ear and which contributes to the character of the complex tone or complex sound. A partial is a physical component of a complex tone.
- **Fundamental Frequency** The fundamental frequency is the frequency component of the lowest frequency in a complex sound.
- **Harmonic** - A harmonic is a partial or overtone whose frequency is an integral multiple of the fundamental tone or fundamental frequency.
  - The 1st overtone is the 2nd harmonic
- **Sub-harmonic** - A sub-harmonic is an integral sub-multiple of the fundamental frequency of the sound to which it is related.

Staff

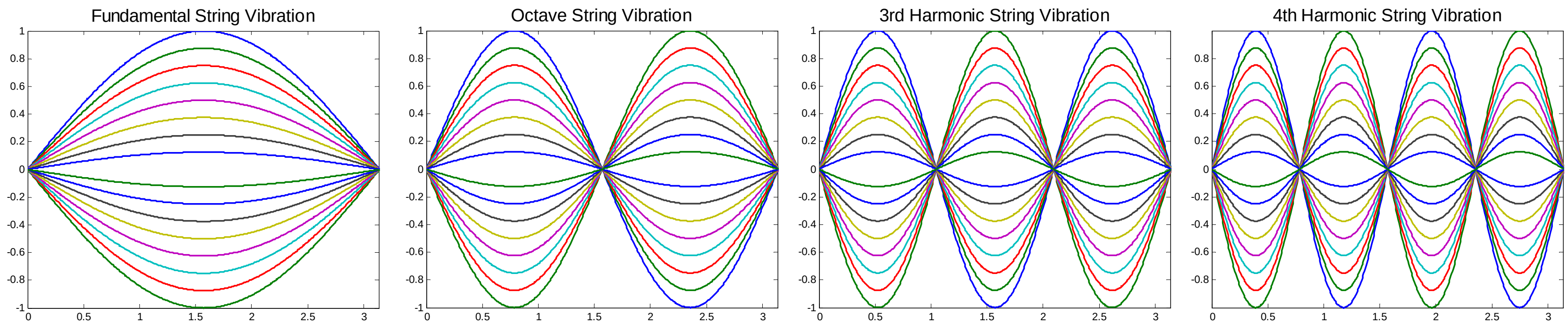


- We need some way of identifying the different notes called C (or D, E, etc)
- There are many methods
  - C0 as middle C,
    - Octaves below are C-1, C-2, etc
    - Octaves above are C1, C2 etc
  - Using upper and lower case a tick marks
- Our preferred method will be to use C4 for middle C.
  - Using this convention C0 is inaudible at below 20Hz, but A0 and B0 are just above that.
  - This corresponds to the convention used in the MIDI standard

Natural origins of scales

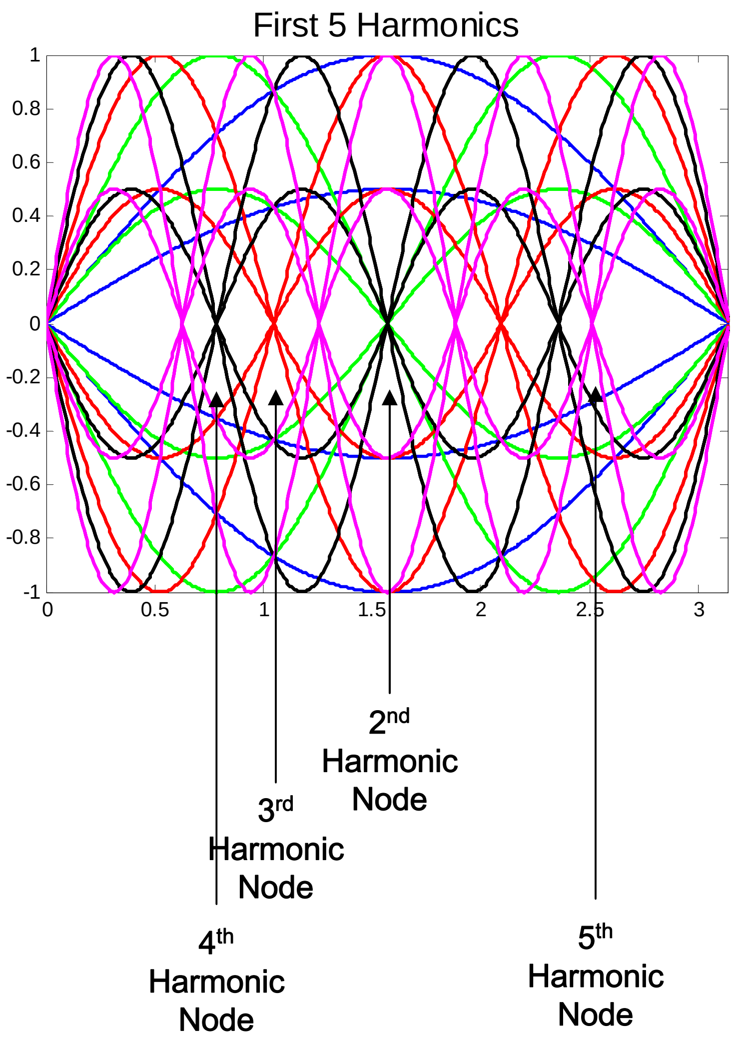
- Scales are supposed to be “smooth, regular, pleasant, and harmonious”
- The Vibrations of the a string create certain harmonics (i.e. overtones that are multiples of the fundamental).
- These are called the “Harmonic Series”
- The intervals are related by the following ratios
  - 2:1, 3:2, 4:3, 5:3, 5:4, 6:5, 8:5, etc

Asside: String Vibrations



- The 4 plots above show the constiuent string vibrations that happen when a string vibrates
- The relative amplitudes of the vibrations depend on
  - The type of excitation
    - Plucked
    - Struck
  - The location of the striking mechanism

Asside: Isolating Harmonics on a vibrating string



- Shown at left are all of the harmonics superimposed.
- It is possible to damped certain harmonics, while letting others ring out.
  - This is called sounding the natural harmonics on a guitar (or other stringed instrument)
  - A guitarist can accomplish this by placing a finger momentarially at the node of a harmonic.
  - All the other harmonics that have a node there as well, will still sound (i.e. the 4th harmonic has a node at the 2nd harmonic's only node)
  - All other harmonics will stop vibrating.

First 10 tones of the Harmonic Series

Frequency (Hz)	Multiple	Ratio	Note	Harmonic	Position
66	1	1:1	C2	1st (fundamental)	2nd line below bass staff
132	2	2:1	C3	2nd (octave)	2nd space in bass staff
198	3	3:2	G3	3rd	Top space in bass staff
264	4	4:3	C4	4th	Middle C – in between staffs
330	5	5:3	E4	5th	Bottom line of treble staff
396	6	6:5	G4	6th	2nd line of treble staff
462	7	7:4	B♭ 4	7th	Middle line of treble staff
528	8	2:1	C5	8th	3rd space from bottom on treble staff
594	9	9:8	D5	9th	4th line from bottom on the treble staff
660	10	5:4	E5	10th	Top space of treble staff

- Notice that the majority of harmonics in the series are
  - Octaves
  - Fifths
  - Thirds
- This might explain the naturally pleasy sound of the major chord

Ratios

- Notice the ratios on the previous slide
  - The ratio of 2:1 is an octave and the most harmonious interval
  - The next most pleasant combination of tones is the fifth, which has a ratios of 3:2
  - After that, the order of pleasing ratios is 2:1 > 3:2 > 4:3 > 5:4 > 6:5 > 8:5 > 5>3
  - There is some subjectivity of what is “pleasing”, but there is also an abundance of musical history based on these ratios
- It can said that the most pleasing ratios are
  - Expressed via two integers
  - Neither should be very large

Intonation

- “Intonation is the process of adjusting or selecting the tones of a musical scale with respect to frequency”
  – i.e. it is the formula and the process for building a musical basis from tones
- A scale that uses only intervals found in the harmonics series is called just intonation.
- Intonating is the process of aligning a fretted instrument so that the fretted notes are correct (according to the rules of equal intonation) relative to the open strings

Cents

$$cents = 1200 * \log_2\left(\frac{x}{y}\right)$$

- Cents are a convenient way to measure the difference between two notes
- As will be seen later, they make the most sense with equal temperament.

Complete Scale of Just Intonation

Interval Name	Frequency ratio of starting point	Cents from starting point	Key of C
Unison	1:1	0	C
Semitone	16:15	111.731	
Minor tone	10:9	182.404	
Major tone	9:8	203.910	D
Minor third	6:5	315.641	
Major third	5:4	386.314	E
Perfect fourth	4:3	498.045	F
Augmented fourth	45:32	590.224	
Diminished fifth	64:45	609.776	
Perfect fifth	3:2	701.955	G
Minor sixth	8:5	813.687	
Major sixth	5:3	884.359	A
Harmonic minor seventh	7:4	968.826	
Grave minor seventh	16:9	996.091	
Minor seventh	9:5	1017.597	
Major seventh	15:8	1088.269	B
Octave	2:1	1200.000	C

Major Scale of Just Intonation

Interval Name	Frequency ratio of starting point	Ratio to previous	Interval to Previous Name	Interval Symbol	Cents from starting point	Key of C
Unison	1:1				0	C
Major tone	9:8	9:8	Major tone	▌▌	203.910	D
Major third	5:4	10:9	Minor tone	▌|	386.314	E
Perfect fourth	4:3	16:15	Semitone	▌	498.045	F
Perfect fifth	3:2	9:8	Major tone	▌▌	701.955	G
Major sixth	5:3	10:9	Minor tone	▌|	884.359	A
Major seventh	15:8	9:8	Major tone	▌▌	1088.269	B
Octave	2:1	16:15	Semitone	▌	1200.000	C

1 ▌▌ 2 ▌| 3 ▌ 4 ▌▌ 5 ▌| 6 ▌▌ 7 ▌ 8

- **Notice the Ratios of the major chords from the scale (C : E : G ; F : A : C and G : B : D)**
  - **The ratios are all 4 : 5 : 6**
- **In total, there are only three distinct ratios needed to build the major scale**
  - **The Major tone (9:8), Minor tone (10:9) and Semitone (16:15)**
- **The lines represent the number of name differences for the notes**
  - **Two lines means 2 notes differences (a whole step)**
  - **One line means 1 notes difference (a half step)**

Minor Scale of Just Intonation

Interval Name	Frequency ratio of starting point	Ratio to previous	Interval to Previous Name	Interval Symbol	Cents from starting point	Key of Am
Unison	1:1				0	A
Major tone	9:8	9:8	Major tone	‖	203.910	B
Minor third	6:5	16:15	Semitone	‖	315.641	C
Perfect fourth	4:3	10:9	Minor tone	❙	498.045	D
Perfect fifth	3:2	9:8	Major tone	‖	701.955	E
Minor sixth	8:5	16:15	Semitone	‖	813.687	F
Minor seventh	9:5	9:8	Major tone	‖	1017.597	G
Octave	2:1	10:9	Minor tone	❙	1200.000	A

1 ‖ 2 ❙ 3 ‖ 4 ‖ 5 ❙ 6 ‖ 7 ‖ 8

- **Notice the Ratios of the minor chords from the scale (A : C : E ; E : G : B and D : F : A)**
  - **The ratios are all 10 : 12 : 15**
- **The same three distinct ratios are needed to build the minor scale**
  - **The Major tone (9:8), Minor tone (10:9) and Semitone (16:15)**
- **The lines represent the number of name differences for the notes**
  - **Two lines means 2 notes differences (a whole step)**
  - **One line means 1 notes difference (a half step)**

Sample Frequencies of Notes in Various Keys

Note	Key														
	C	**D**	**E**	**F♯**	**G**	**A**	**B**	**C♯**	**D♭**	**E♭**	**F**	**G♭**	**A♭**	**B♭**	**C♭**
C♭4											247.2				247.2
C4	264.0				264.0				260.7	260.7	264.0		260.7	264.0	
C♯4		278.4	275.0	278.4		275	278.4	278.4							
D♭4									278.1			278.1	278.1		278.1
D4	297.0	297.0			297.0	293.3				293.3	293.3				
D♯4			309.4	309.4			309.4	313.2							

- **This table shows the calculations for some notes in some keys (all using A=440 Hz and the ratios shown previously)**
- **Note that the same note may have different frequencies in different keys**
  - **This is one of the fundamental problems that led to the abandonment of just temperament**
  - **This also made transposition from one key to another difficult.**
- **Note that the enharmonics that we considered equivalent last week are different frequencies**
- **It is possible to complete the full table for all keys**
  - **Doing this would show a requirement of at least 30 discrete frequencies per octave**

Equal Tempered Pitch

- "Temperament is the process of reducing the number of tones per octave by altering the frequency of the tones from the exact frequencies of just intonation"
- The solution is to divide the octave into 12 equal steps such that the ratios follow the following pattern

$$1, f, f^2, f^3, f^4, f^5, f^6, f^7, f^8, f^9, f^{10}, f^{11}, f^{12}$$

- Subject to the following condition on the octave

$$f^{12} = 2$$

- Means that

$$f = \sqrt[12]{2}$$

Intervals in Equal Temperment

- This divides the octave into 12 equal tempered half tones or half step. Each interval is computed as a multiple of the twelfth root of 2.
- Two half steps or half tones equals one whole step or whole tone.
- Each half tone is exactly 100 cents
- Given this formula one could create a table of equal temperament showing the frequency for each note name
  - This would show that each note name has exactly one frequency
  - There are only 12 distinct notes between octaves

Complete Scale of Equal Intonation

Interval Name	Frequency ratio of starting point	Cents from starting point	Key of C
Unison	1:1	0	C
Semitone (Minor second)	$\sqrt[12]{2}:1$	100	
Whole tone (Major second)	$\sqrt[12]{2^2}:1$	200	D
Minor third	$\sqrt[12]{2^3}:1$	300	
Major third	$\sqrt[12]{2^4}:1$	400	E
Perfect fourth	$\sqrt[12]{2^5}:1$	500	F
Augmented fourth / Diminished fifth	$\sqrt[12]{2^6}:1$	600	
Perfect fifth	$\sqrt[12]{2^7}:1$	700	G
Minor sixth	$\sqrt[12]{2^8}:1$	800	
Major sixth	$\sqrt[12]{2^9}:1$	900	A
Minor seventh	$\sqrt[12]{2^{10}}:1$	1000	
Major seventh	$\sqrt[12]{2^{11}}:1$	1100	B
Octave	$\sqrt[12]{2^{12}}:1$	1200	C

Major Scale of Equal Intonation

Interval Name	Frequency ratio of starting point	Ratio to previous	Interval to Previous Name	Interval Symbol	Cents from starting point	Key of C
Unison	1:1				0	C
Major second	$\sqrt[12]{2^2}:1$	$\sqrt[12]{2^2}:1$	Whole step	▌▌	200	D
Major third	$\sqrt[12]{2^4}:1$	$\sqrt[12]{2^2}:1$	Whole step	▌▌	400	E
Perfect fourth	$\sqrt[12]{2^5}:1$	$\sqrt[12]{2}:1$	Half step	▌	500	F
Perfect fifth	$\sqrt[12]{2^7}:1$	$\sqrt[12]{2^2}:1$	Whole step	▌▌	700	G
Major sixth	$\sqrt[12]{2^9}:1$	$\sqrt[12]{2^2}:1$	Whole step	▌▌	900	A
Major seventh	$\sqrt[12]{2^{11}}:1$	$\sqrt[12]{2^2}:1$	Whole step	▌▌	1100	B
Octave	2:1	$\sqrt[12]{2}:1$	Half step	▌	1200	C

1 ▌▌ 2 ▌▌ 3 ▌ 4 ▌▌ 5 ▌▌ 6 ▌▌ 7 ▌ 8

- **The same three distinct ratios are needed to build the major scale**
  - **The whole step and the half step.**

Minor Scale of Equal Intonation

Interval Name	Frequency ratio of starting point	Ratio to previous	Interval to Previous Name	Interval Symbol	Cents from starting point	Key of Am
Unison	1:1				0	A
Major second	$\sqrt[12]{2^2}:1$	$\sqrt[12]{2^2}:1$	Whole step	▌▌	200	B
Minor third	$\sqrt[12]{2^3}:1$	$\sqrt[12]{2}:1$	Half step	▌	300	C
Perfect fourth	$\sqrt[12]{2^5}:1$	$\sqrt[12]{2^2}:1$	Whole step	▌▌	500	D
Perfect fifth	$\sqrt[12]{2^7}:1$	$\sqrt[12]{2^2}:1$	Whole step	▌▌	700	E
Minor sixth	$\sqrt[12]{2^8}:1$	$\sqrt[12]{2}:1$	Half step	▌	800	F
Minor seventh	$\sqrt[12]{2^{10}}:1$	$\sqrt[12]{2^2}:1$	Whole step	▌▌	1000	G
Octave	2:1	$\sqrt[12]{2^2}:1$	Whole step	▌▌	1200	A

- **The Natural Minor Scale can be computed as follows:**
  1 ▌▌ 2 ▌ 3 ▌▌ 4 ▌▌ 5 ▌ 6 ▌▌ 7 ▌▌ 8
- **The Harmonic Minor Scale can be computes as follows:**
  1 ▌▌ 2 ▌ 3 ▌▌ 4 ▌▌ 5 ▌ 6 ▌▌ 7 ▌▌▌ 8
- **The Melodic Minor Scale can be computed as follows:**
  - **Ascending**
    1 ▌▌ 2 ▌ 3 ▌▌ 4 ▌▌ 5 ▌▌ 6 ▌▌ 7 ▌ 8
  - **Descending**
    1 ▌▌ 2 ▌ 3 ▌▌ 4 ▌▌ 5 ▌▌ 6 ▌▌ 7 ▌▌ 8

Comparison

Equal Temperament Interval Name	Frequency ratio of starting point	Cents from starting point	Just Temperament Interval Name	Frequency Ratio from Starting Point	Cents from starting point	Key of C
Unison	1:1	0	1:1		0	C
Semitone (Minor second)	$\sqrt[12]{2}:1$	100	Semitone	16:15	111.731	
Whole tone (Major second)	$\sqrt[12]{2^2}:1$	200	Major tone	9:8	203.910	D
Minor third	$\sqrt[12]{2^3}:1$	300	Minor third	6:5	315.641	
Major third	$\sqrt[12]{2^4}:1$	400	Major third	5:4	386.314	E
Perfect fourth	$\sqrt[12]{2^5}:1$	500	Perfect fourth	4:3	498.045	F
Augmented fourth / Diminished fifth	$\sqrt[12]{2^6}:1$	600	Diminished fifth	64:45	609.776	
Perfect fifth	$\sqrt[12]{2^7}:1$	700	Perfect fifth	3:2	701.955	G
Minor sixth	$\sqrt[12]{2^8}:1$	800	Minor sixth	8:5	813.687	
Major sixth	$\sqrt[12]{2^9}:1$	900	Major sixth	5:3	884.359	A
Minor seventh	$\sqrt[12]{2^{10}}:1$	1000	Minor seventh	9:5	1017.597	
Major seventh	$\sqrt[12]{2^{11}}:1$	1100	Major seventh	15:8	1088.269	B
Octave	$\sqrt[12]{2^{12}}:1$	1200	Octave	2:1	1200.000	C

Results

- The maximum definition between just and equal temperament is +/- 2 percent
  - Good enough for most cases
  - Some things are still tweaked a bit
    - Stretch tuning
    - Micro tonal adjustmens

Annotated References

- Music, Physics & Engineering, Olson
- Jeans
- http://www.harmony-central.com/Guitar/harmonics.html
  - Nice explanation of vibrating string harmonics
- http://en.wikipedia.org/wiki/Meantone_temperament
  - More excellent wikipedia articles on temperament
- http://www.phys.unsw.edu.au/jw/notes.html
  - MIDI reference chart